

C_n -MOVES AND THE DIFFERENCE OF JONES POLYNOMIALS FOR LINKS

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ABSTRACT. The Jones polynomial $V_L(t)$ for an oriented link L is a one-variable Laurent polynomial link invariant discovered by Jones. For any integer $n \geq 3$, we show that: (1) the difference of Jones polynomials for two oriented links which are C_n -equivalent is divisible by $(t-1)^n (t^2+t+1) (t^2+1)$, and (2) there exists a pair of two oriented knots which are C_n -equivalent such that the difference of the Jones polynomials for them equals $(t-1)^n (t^2+t+1) (t^2+1)$.

1. INTRODUCTION

The *Jones polynomial* $V_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$ is an integral Laurent polynomial link invariant for an oriented link L defined by the following formulae:

$$\begin{aligned} V_O(t) &= 1, \\ t^{-1}V_{L_+}(t) - tV_{L_-}(t) &= \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V_{L_0}(t), \end{aligned}$$

where O denotes the trivial knot and L_+ , L_- and L_0 are oriented links which are identical except inside the depicted regions as illustrated in Fig. 1.1 [6]. The triple of oriented links (L_+, L_-, L_0) is called a *skein triple*. Jones also showed the following property of the Jones polynomials for oriented knots.

Theorem 1.1. (Jones [6, Proposition 12.5]) *For any two oriented knots J and K , $V_J(t) - V_K(t)$ is divisible by $(t-1)^2 (t^2+t+1)$.*

On the basis of Theorem 1.1, for an oriented knot K , Jones called the polynomial $W_K(t) = \{1 - V_K(t)\} / (t-1)^2 (t^2+t+1)$ a *simplified polynomial* and made a table of the simplified polynomials for knots up to 10 crossings [6]. In particular, if K is the right-handed trefoil knot then $W_K(t) = 1$. So the polynomial $(t-1)^2 (t^2+t+1)$ is maximal as a divisor of the difference of Jones polynomials of any pair of two oriented knots.

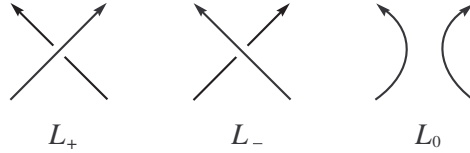


FIGURE 1.1. Skein triple (L_+, L_-, L_0)

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Our purpose in this paper is to examine the difference of Jones polynomials for two oriented links which are C_n -equivalent, where a C_n -equivalence is an equivalence relation on oriented links introduced by Habiro [8] and Gusarov [5] independently as follows. For a positive integer n , a C_n -move is a local move on oriented links as illustrated in Fig. 1.2 if $n \geq 2$, and a C_1 -move is a crossing change. Two oriented links are said to be C_n -equivalent if they are transformed into each other by C_n -moves and ambient isotopies. By the definition of a C_n -move, it is easy to see that a C_n -equivalence implies a C_{n-1} -equivalence. Note that a C_2 -move equals a *delta move* introduced by Matveev [17] and Murakami-Nakanishi [20] independently as illustrated in Fig. 1.3 (1), and a C_3 -move equals a *clasp-pass move* introduced by Habiro [7] as illustrated in Fig. 1.3 (2). A C_n -move is closely related to the *Vassiliev invariants* of oriented links [26], [2], [1], [25]. It is known that if two oriented links are C_n -equivalent then they have the same Vassiliev invariants of order $\leq n-1$, and specially for oriented knots, the converse is also true [9], [5].

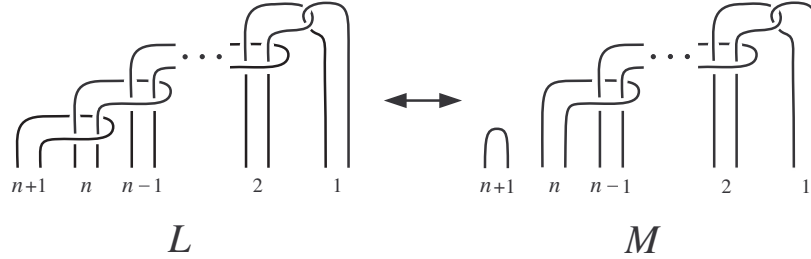


FIGURE 1.2. C_n -move ($n \geq 2$)

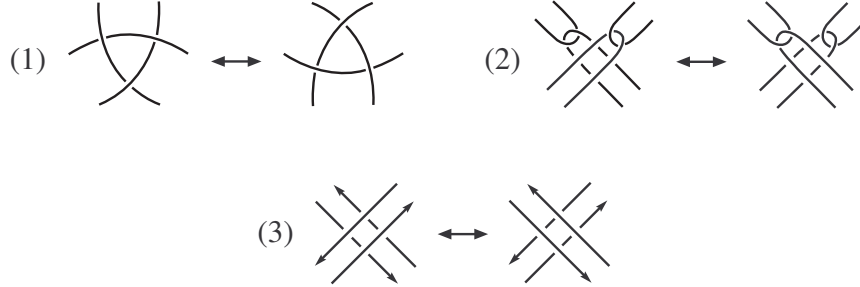


FIGURE 1.3. (1) Delta move, (2) Clasp-pass move, (3) Pass move

Now let us generalize Theorem 1.1 to oriented links which are C_n -equivalent.

Theorem 1.2. (1) If two oriented links L and M are C_2 -equivalent, then $V_L(t) - V_M(t)$ is divisible by $(t-1)^2(t^2+t+1)$.
(2) For any integer $n \geq 3$, if two oriented links L and M are C_n -equivalent, then $V_L(t) - V_M(t)$ is divisible by $(t-1)^n(t^2+t+1)(t^2+1)$.

We remark that Theorem 1.2 (1) was also observed in [4, Theorem 2] for oriented knots by using the Kauffman bracket. Since any two oriented knots are C_2 -equivalent [20], Theorem 1.1 is deduced from Theorem 1.2 (1).

In the case of $n \geq 3$, we show the maximality of $(t-1)^n (t^2+t+1) (t^2+1)$ as a divisor of the difference of Jones polynomials for oriented links which are C_n -equivalent as follows. Let J_n and K_n be two oriented knots as illustrated in Fig. 1.4. Note that J_n and K_n are transformed into each other by a single C_n -move, see Fig. 1.5. Then we have the following.

Theorem 1.3.

$$V_{J_n}(t) - V_{K_n}(t) = (-1)^{n+1} (t-1)^n (t^2+t+1) (t^2+1).$$

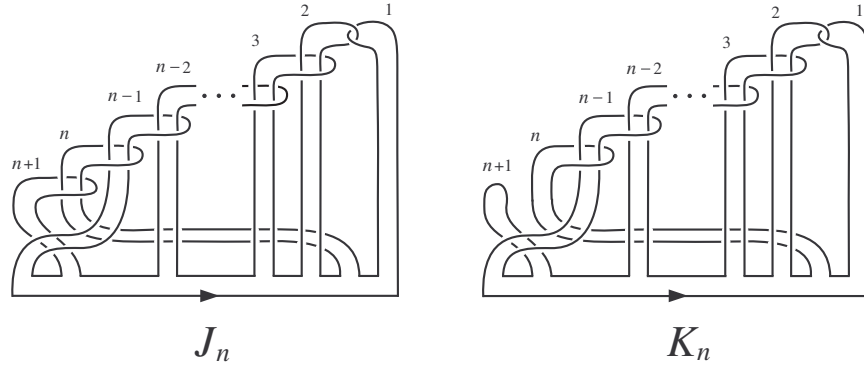


FIGURE 1.4. Oriented knots J_n and K_n ($n \geq 3$)

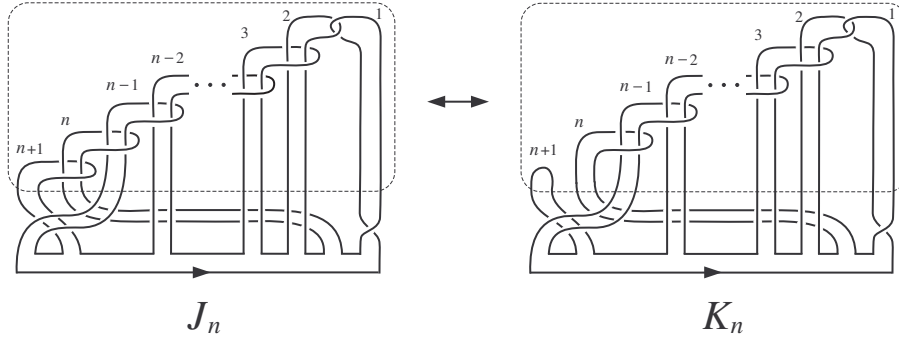


FIGURE 1.5. J_n and K_n are transformed into each other by a single C_n -move

In section 2, we prove Theorem 1.2 and give its applications to the study of the difference of Vassiliev invariants of order $\leq n$ for two oriented links which are C_n -equivalent. In section 3, we prove Theorem 1.3 without knowing $V_{J_n}(t)$ and $V_{K_n}(t)$ individually by applying Kanenobu's formula for the difference of Jones polynomials for two oriented knots which are transformed into each other by a single C_n -move (Lemma 3.1) and a C_n -move which does not change the knot type (Lemma 3.2).

2. PROOF OF THEOREM 1.2

We recall the following results about the special values of the Jones polynomial. Here an r -component oriented link L is said to be *proper* if $\text{lk}(K, L \setminus K) \equiv 0 \pmod{2}$ for each component K of L , where lk denotes the linking number, and the *Arf invariant* is a link invariant introduced in [24] defined for only proper links.

Lemma 2.1. *Let L be an r -component oriented link. Then the followings holds.*

- (1) ([6, (12.1)]) $V_L(1) = (-2)^{r-1}$.
- (2) ([6, (12.4)]) $V_L(e^{2\pi\sqrt{-1}/3}) = (-1)^{r-1}$.
- (3) (Murakami [19]) $V_L(\sqrt{-1}) = (\sqrt{-2})^{r-1} \cdot (-1)^{\text{Arf}(L)}$ if L is proper, and 0 if L is nonproper, where Arf denotes the Arf invariant.

For an oriented link L , we denote the l -th derivative at 1 of the Jones polynomial $V_L(t)$ by $V_L^{(l)}(1)$. It is known that $V_L^{(l)}(1)$ is a Vassiliev invariant of order $\leq l$ [12]. Then we have the following.

Lemma 2.2. *Let L and M be two oriented r -component links and n an integer with $n \geq 2$. Then $V_L^{(l)}(1) = V_M^{(l)}(1)$ for $l = 1, 2, \dots, n-1$ if and only if $V_L(t) - V_M(t)$ is divisible by $(t-1)^n(t^2+t+1)$.*

Proof. The ‘if’ part is clear because $(t-1)^n$ divides $V_L(t) - V_M(t)$. We show the ‘only if’ part by the induction on n . Assume that $n = 2$. By Lemma 2.1 (1) and (2), there exists a polynomial $g(t) \in \mathbb{Z}[t^{\pm 1/2}]$ such that

$$(2.1) \quad V_L(t) - V_M(t) = (t^3 - 1)g(t).$$

Then by differentiating both sides in (2.1), we have

$$(2.2) \quad V_L^{(1)}(t) - V_M^{(1)}(t) = 3t^2g(t) + (t^3 - 1)g^{(1)}(t).$$

Thus by the assumption and (2.2), we have $g(1) = 0$. This implies that $V_L(t) - V_M(t)$ is divisible by $(t-1)^2(t^2+t+1)$. Next assume that $n \geq 3$ and $V_L^{(l)}(1) = V_M^{(l)}(1)$ for $l = 0, 1, \dots, n-1$. By the induction hypothesis, it follows that there exists a polynomial $h(t) \in \mathbb{Z}[t^{\pm 1/2}]$ such that

$$(2.3) \quad V_L(t) - V_M(t) = (t-1)^{n-1}(t^2+t+1)h(t).$$

Let us denote $(t^2+t+1)h(t)$ by $\tilde{h}(t)$. Then by (2.3) and the assumption, we have

$$(2.4) \quad 0 = V_L^{(n-1)}(1) - V_M^{(n-1)}(1) = (n-1)! \tilde{h}(1).$$

Thus we have $0 = \tilde{h}(1) = 3h(1)$, namely $h(1) = 0$. This implies that $t-1$ divides $h(t)$, therefore we have the desired conclusion. \square

Remark 2.3. For an r -component oriented link L , it is known that

$$V_L^{(1)}(1) = -3(-2)^{r-2}\text{Lk}(L)$$

if $r \geq 2$ and 0 if $r = 1$, where Lk denotes the total linking number, that is the summation of all pairwise linking numbers of L [6, (12.2)]. Thus Lemma 2.2 implies Theorem 1.1 in case $r = 1$, and $\text{Lk}(L) = \text{Lk}(M)$ if and only if $V_L(t) - V_M(t)$ is divisible by $(t-1)^2(t^2+t+1)$ in case $r \geq 2$.

Lemma 2.4. *For an integer $n \geq 3$, if two oriented links L and M are C_n -equivalent, then $V_L(t) - V_M(t)$ is divisible by $t^2 + 1$.*

Proof. Let L and M be two r -component oriented links which are C_n -equivalent. Then L and M are C_3 -equivalent. Note that a C_3 -move = a clasp-pass move can be realized by a single *pass move* [13] as illustrated in Fig. 1.3 (3), and a pass move does not change the Arf invariant of a proper link [20, Appendix]. If L is proper, then M is also proper because L and M also are C_2 -equivalent and a C_2 -move does not change the pairwise linking numbers. Then by Lemma 2.1 (3), we have $V_L(\sqrt{-1}) = V_M(\sqrt{-1})$. If L is nonproper, then M is also nonproper and by Lemma 2.1 (3), we have $V_L(\sqrt{-1}) = 0 = V_M(\sqrt{-1})$. \square

Proof of Theorem 1.2. As we mentioned before, if two oriented r -component links L and M are C_n -equivalent then $V_L^{(l)}(1) = V_M^{(l)}(1)$ for $l = 1, 2, \dots, n-1$. By combining this fact with Lemma 2.2, we have (1) in case $n = 2$, and by combining this fact with Lemma 2.2 and Lemma 2.4, we have (2). \square

As an application, we give alternative short proofs for two theorems shown by H. A. Miyazawa. Note that these theorems were proved by fairly combinatorial argument, that is, by making up a list of oriented C_n -moves carefully and checking the congruence for each of the cases. First we show the following as a direct consequence of Theorem 1.2 (2).

Theorem 2.5. (H. A. Miyazawa [18, Theorem 1.5]) *For an integer $n \geq 3$, if two oriented links L and M are C_n -equivalent, then it follows that*

$$V_L^{(n)}(1) \equiv V_M^{(n)}(1) \pmod{6 \cdot n!}.$$

Proof. Assume that two oriented links L and M are C_n -equivalent. Then by Theorem 1.2 (2), there exists a polynomial $f(t) \in \mathbb{Z}[t^{\pm 1/2}]$ such that

$$(2.5) \quad V_L(t) - V_M(t) = (t-1)^n (t^2 + t + 1) (t^2 + 1) f(t).$$

Let us denote $(t^2 + t + 1) (t^2 + 1) f(t)$ by $\tilde{f}(t)$. Then by (2.5), we have

$$V_L^{(n)}(1) - V_M^{(n)}(1) = n! \cdot \tilde{f}(1) = n! \cdot 6f(1).$$

Thus we have the result. \square

Miyazawa also showed the best possibility of Theorem 2.5 by exhibiting two pairs of two oriented knots which are C_n -equivalent whose differences of the Jones polynomials do not equal $6 \cdot n!$ but the greatest common divisor of them is $6 \cdot n!$. The best possibility of Theorem 2.5 may also be given by two oriented knots J_n and K_n in Theorem 1.3 whose difference of the Jones polynomials exactly equals $6 \cdot n!$. Such an example was also observed by Horiuchi [10].

On the other hand, the *Conway polynomial* $\nabla_L(z) \in \mathbb{Z}[z]$ is an integral polynomial link invariant for an oriented link L defined by the following formulae:

$$\begin{aligned} \nabla_O(z) &= 1, \\ \nabla_{L_+}(z) - \nabla_{L_-}(z) &= z \nabla_{L_0}(z), \end{aligned}$$

where (L_+, L_-, L_0) is a skein triple in Fig. 1.1 [3]. Note that

$$(2.6) \quad V_L(-1) = \nabla_L(-2\sqrt{-1})$$

and the absolute value of $V_L(-1)$ is known as the determinant of L . We denote the coefficient of z^l in $\nabla_L(z)$ by $a_l(L)$. Then it is known that the Conway polynomial

of an r -component oriented link L is of the following form

$$(2.7) \quad \nabla_L(z) = \sum_{i \geq 0} a_{r+2i-1}(L) z^{r+2i-1}.$$

It is known that $a_l(L)$ is a Vassiliev invariant of order $\leq l$ [1]. Thus if two oriented links L and M are C_n -equivalent, then $a_l(L) = a_l(M)$ for $l \leq n-1$. In the case of $l = n$, Miyazawa showed the following. Note that in the case of oriented knots, this had been obtained by Ohyama-Ogushi [21].

Theorem 2.6. (H. A. Miyazawa [18, Theorem 1.3]) *For an integer $n \geq 3$, if two oriented links L and M are C_n -equivalent, then it follows that*

$$a_n(L) \equiv a_n(M) \pmod{2}.$$

Proof. Let L and M be two r -component oriented links which are C_n -equivalent. If $n \equiv r \pmod{2}$, then by (2.7) we have $a_n(L) = a_n(M) = 0$. Assume that $n \not\equiv r \pmod{2}$. Then by Theorem 1.2 (2) and (2.6), there exists a polynomial $W(t) \in \mathbb{Z}[t^{\pm 1/2}]$ such that

$$\begin{aligned} (-1)^n \cdot 2^{n+1} \cdot W(-1) &= V_L(-1) - V_M(-1) \\ &= \nabla_L(-2\sqrt{-1}) - \nabla_M(-2\sqrt{-1}) \\ &= \sum_{i \geq 1} \{a_{n+2i-2}(L) - a_{n+2i-2}(M)\} \cdot (-2\sqrt{-1})^{n+2i-2}. \end{aligned}$$

This implies

$$0 \equiv \{a_n(L) - a_n(M)\} \cdot 2^n \pmod{2^{n+1}}$$

and therefore $a_n(L) - a_n(M)$ must be even. \square

Miyazawa showed that Theorem 2.6 is also best possible. Furthermore, Ohyama-Yamada proved that for an integer $n \geq 2$, if two oriented knots J and K are transformed into each other by a single C_{2n} -move then $a_{2n}(J) - a_{2n}(K) = 0, \pm 2$ [23, Theorem 1.3].

3. PROOF OF THEOREM 1.3

We show three lemmas needed to prove the Theorem 1.3. The first lemma is Kanenobu's formula for the difference of Jones polynomials for two oriented links which are transformed into each other by a single C_n -move. Let L and M be two oriented links which are transformed into each other by a single C_n -move as illustrated in Fig. 1.2. Let c_{j1}, c_{j2} ($j = 2, 3, \dots, n$) and c_1 be crossings of L as illustrated in Fig. 3.1. We denote the sign of c_1 by ε_1 and the sign of c_{j1} by ε_j ($j = 2, 3, \dots, n$). Let $L[\delta_2, \delta_3, \dots, \delta_n]$ be the link obtained from L by smoothing the crossing c_1 , smoothing the crossing c_{j1} if $\delta_j = 1$, and changing the crossing c_{j1} and smoothing the crossing c_{j2} if $\delta_j = -1$ ($j = 2, 3, \dots, n$). Then the following formula holds.

Lemma 3.1. (Kanenobu [11, (4.10)])

$$\begin{aligned} &V_L(t) - V_M(t) \\ &= \left(\prod_{i=1}^n \varepsilon_i \right) t^{\sum_{i=1}^n \varepsilon_i - \frac{n}{2}} (t-1)^n \sum_{\delta_2, \delta_3, \dots, \delta_n = \pm 1} \left(\prod_{j=2}^n \delta_j \right) V_{L[\delta_2, \delta_3, \dots, \delta_n]}(t). \end{aligned}$$

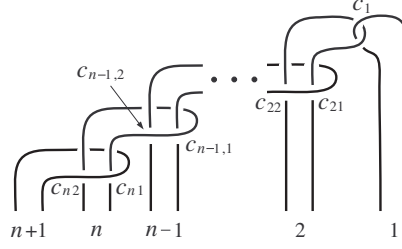


FIGURE 3.1. Crossings c_1 , c_{j1} and c_{j2} of L ($j = 2, 3, \dots, n$)

Next we show the second lemma. For an integer $n \geq 3$, let L'_n and M'_n be two links as illustrated in Fig. 3.2, where T is an arbitrary 2-string tangle which are same for both links. Note that L'_n and M'_n are transformed into each other by a single C_n -move. Then we have the following.

Lemma 3.2. L'_n and M'_n are ambient isotopic.

Proof. See Fig. 3.3. □

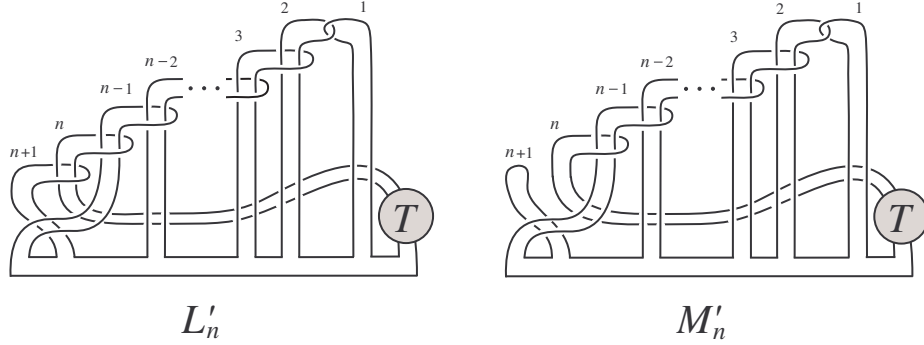


FIGURE 3.2. Two links L'_n and M'_n ($n \geq 3$)

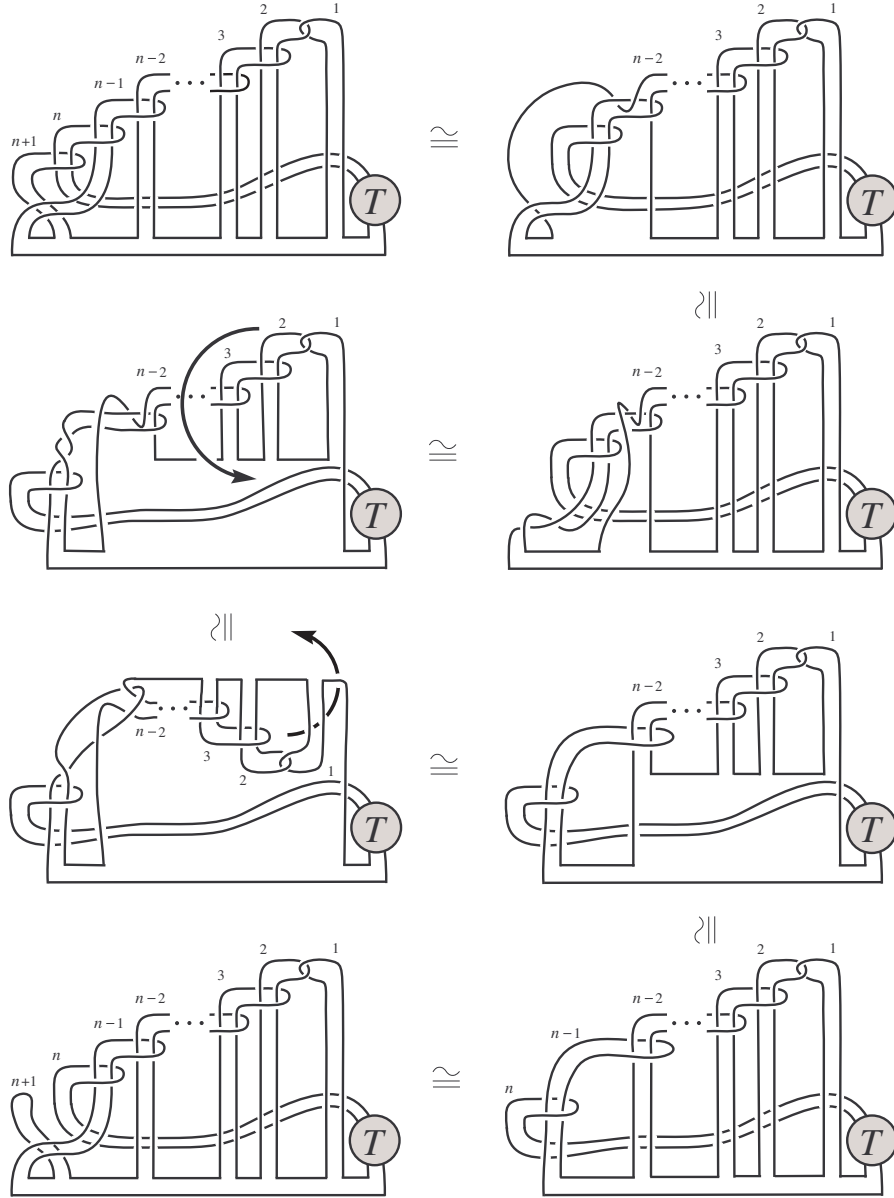
Lemma 3.2 gives a new example of a C_n -move which does not change the knot type. Such an example was first discovered by Ohyama-Tsukamoto [22].

The third lemma is a calculation of the Jones polynomial for a $(2, -m)$ -torus knot or link N_m for a non-negative integer m as illustrated in Fig. 3.6 (1). Note that such a calculation has been already known, see [14, pp. 37], [16, Lemma 2.1] for example. However we state a formula and give a proof for reader's convenience.

Lemma 3.3.

$$V_{N_m}(t) = \left(-t^{-\frac{1}{2}}\right)^m \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) + \frac{\left(-t^{\frac{1-3m}{2}}\right) \{1 - (-t)^m\}}{1+t}.$$

Proof. Note that N_0 is the trivial 2-component link and N_1 is the trivial knot. Then we can check the formula directly for $m = 0, 1$. Assume that $m \geq 2$. Then

FIGURE 3.3. L'_n and M'_n are ambient isotopic

we obtain the skein triple (N_{m-2}, N_m, N_{m-1}) easily and therefore we have

$$(3.1) \quad t^{-1}V_{N_{m-2}}(t) - tV_{N_m}(t) = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V_{N_{m-1}}(t).$$

By (3.1), we have

$$\begin{aligned}
 (3.2) \quad V_{N_m}(t) + t^{-\frac{1}{2}} V_{N_{m-1}}(t) &= t^{-\frac{3}{2}} \left\{ V_{N_{m-1}}(t) + t^{-\frac{1}{2}} V_{N_{m-2}}(t) \right\} \\
 &= \left(t^{-\frac{3}{2}} \right)^{m-1} \left\{ V_{N_1}(t) + t^{-\frac{1}{2}} V_{N_0}(t) \right\} \\
 &= -t^{\frac{1-3m}{2}}.
 \end{aligned}$$

Then by (3.2), we have

$$\begin{aligned}
 V_{N_m}(t) &= -t^{-\frac{1}{2}} V_{N_{m-1}}(t) - t^{\frac{1-3m}{2}} \\
 &= \left(-t^{-\frac{1}{2}} \right)^m V_{N_0}(t) + \sum_{i=1}^m \left(-t^{\frac{1-3m}{2}} \right) (-t)^{i-1} \\
 &= \left(-t^{-\frac{1}{2}} \right)^m \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) + \frac{\left(-t^{\frac{1-3m}{2}} \right) \{1 - (-t)^m\}}{1+t}.
 \end{aligned}$$

□

Proof of Theorem 1.3. First we show in the case of $n = 3$ and 4. If $n = 3$, by a calculation (with the help of [15]) we have

$$\begin{aligned}
 V_{J_3}(t) &= t^{-1} - 2 + 4t - 4t^2 + 5t^3 - 5t^4 + 3t^5 - 2t^6 + t^7, \\
 V_{K_3}(t) &= t^{-1} - 1 + 2t - 2t^2 + 2t^3 - 2t^4 + t^5.
 \end{aligned}$$

Then we have

$$V_{J_3}(t) - V_{K_3}(t) = (t-1)^3 (t^2 + t + 1) (t^2 + 1).$$

If $n = 4$, by a calculation we have

$$\begin{aligned}
 V_{J_4}(t) &= -t^{-2} + 4t^{-1} - 8 + 13t - 15t^2 + 17t^3 - 16t^4 + 12t^5 - 8t^6 + 4t^7 - t^8, \\
 V_{K_4}(t) &= -t^{-2} + 4t^{-1} - 7 + 10t - 11t^2 + 12t^3 - 10t^4 + 7t^5 - 4t^6 + t^7.
 \end{aligned}$$

Then we have

$$V_{J_4}(t) - V_{K_4}(t) = -(t-1)^4 (t^2 + t + 1) (t^2 + 1).$$

From now on, we assume that $n \geq 5$. Since $\varepsilon_i = 1$ for any i , we have

$$(3.3) \quad V_{J_n}(t) - V_{K_n}(t) = t^{\frac{n}{2}} (t-1)^n \sum_{\delta_2, \dots, \delta_n = \pm 1} \left(\prod_{j=2}^n \delta_j \right) V_{J_n[\delta_2, \dots, \delta_n]}(t).$$

If $\delta_2 = -1$, we can see that $J_n[-1, \delta_3, \dots, \delta_{n-1}, 1]$ and $J_n[-1, \delta_3, \dots, \delta_{n-1}, -1]$ are ambient isotopic, see Fig. 3.4. Thus by (3.3), we have

$$(3.4) \quad V_{J_n}(t) - V_{K_n}(t) = t^{\frac{n}{2}} (t-1)^n \sum_{\delta_3, \dots, \delta_n = \pm 1} \left(\prod_{j=3}^n \delta_j \right) V_{J_n[1, \delta_3, \dots, \delta_n]}(t).$$

Let k be an integer satisfying $3 \leq k \leq n-2$. Note that k is also satisfied with $3 \leq n-k+1 \leq n-2$. Then we can see that $J_n[1, \dots, 1, -1, \delta_{k+1}, \dots, \delta_n]$ is ambient isotopic to $L'_{n-k+1}[\delta_{k+1}, \dots, \delta_n]$ for some 2-string tangle T_k , see Fig. 3.5, where

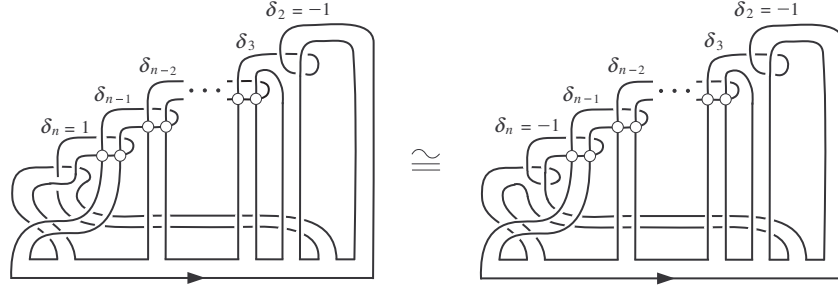


FIGURE 3.4. $J_n[-1, \delta_3, \dots, \delta_{n-1}, 1]$ and $J_n[-1, \delta_3, \dots, \delta_{n-1}, -1]$ are ambient isotopic

L'_{n-k+1} and M'_{n-k+1} are corresponding knots as illustrated in Fig. 3.2. Then by Lemma 3.2, we have L'_{n-k+1} and M'_{n-k+1} are ambient isotopic and therefore

$$\begin{aligned}
 (3.5) \quad & \sum_{\delta_{k+1}, \dots, \delta_n = \pm 1} \left(\prod_{j=k+1}^n \delta_j \right) V_{J_n[1, \dots, 1, -1, \delta_{k+1}, \dots, \delta_n]}(t) \\
 &= \left\{ V_{L'_{n-k+1}}(t) - V_{M'_{n-k+1}}(t) \right\} / \left(- \prod_{i=k+1}^n \varepsilon_i \right) t^{-1 + \sum_{i=k+1}^n \varepsilon_i - \frac{1}{2}(n-k)} (t-1)^{n-k} \\
 &= 0.
 \end{aligned}$$

Thus by (3.4) and (3.5), we have

$$(3.6) \quad V_{J_n}(t) - V_{K_n}(t) = t^{\frac{n}{2}} (t-1)^n \sum_{\delta_{n-1}, \delta_n = \pm 1} \delta_{n-1} \delta_n V_{J_n[1, \dots, 1, \delta_{n-1}, \delta_n]}(t).$$

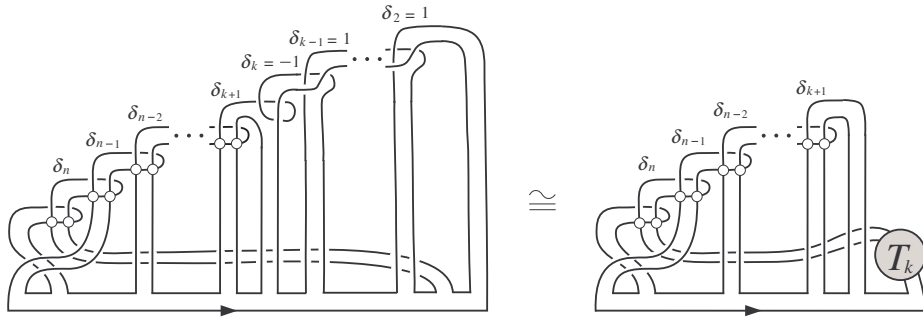


FIGURE 3.5. $J_n[1, \dots, 1, -1, \delta_{k+1}, \dots, \delta_n]$ is ambient isotopic to $L'_{n-k+1}[\delta_{k+1}, \dots, \delta_n]$ for some 2-string tangle T_k

We can see easily that $J_n[1, \dots, 1, -1, -1]$ is ambient isotopic to N_{n-3} , $J_n[1, \dots, 1, -1, 1]$ is ambient isotopic to the split union of N_{n-4} and the trivial knot, and $J_n[1, \dots, 1, -1]$ is ambient isotopic to the connected sum of N_{n-3} , the Hopf link with linking number

1 and the Hopf link with linking number -1 . Thus we have

$$(3.7) \quad V_{J_n[1, \dots, 1, -1, -1]}(t) = V_{N_{n-3}}(t),$$

$$(3.8) \quad V_{J_n[1, \dots, 1, -1, 1]}(t) = \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V_{N_{n-4}}(t),$$

$$(3.9) \quad V_{J_n[1, \dots, 1, -1]}(t) = \left(-t^{\frac{5}{2}} - t^{\frac{1}{2}}\right) \left(-t^{-\frac{5}{2}} - t^{-\frac{1}{2}}\right) V_{N_{n-3}}(t).$$

Further, $J_n[1, \dots, 1]$ is ambient isotopic to the oriented link as illustrated in Fig. 3.6 (2), where $m = n - 4$. We obtain the skein triple $(N_{n-3}, J_n[1, \dots, 1], N_{n-4})$ by changing and smoothing the marked crossing in Fig. 3.6. Thus we have

$$(3.10) \quad V_{J_n[1, \dots, 1]}(t) = t^{-2} V_{N_{n-3}}(t) - t^{-1} \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) V_{N_{n-4}}(t).$$

By combining with (3.6), (3.7), (3.8), (3.9) and (3.10), we have

$$(3.11) \quad \begin{aligned} & V_{J_n}(t) - V_{K_n}(t) \\ &= t^{\frac{n}{2}} (t-1)^n \left\{ (-1-t^2) V_{N_{n-3}}(t) + \left(t^{\frac{1}{2}} + t^{-\frac{3}{2}}\right) V_{N_{n-4}}(t) \right\} \\ &= t^{\frac{n}{2}} (t-1)^n (1+t^2) \left\{ -V_{N_{n-3}}(t) + t^{-\frac{3}{2}} V_{N_{n-4}}(t) \right\}. \end{aligned}$$

Here, by Lemma 3.3, we also have

$$(3.12) \quad \begin{aligned} & -V_{N_{n-3}}(t) + t^{-\frac{3}{2}} V_{N_{n-4}}(t) \\ &= -\left(-t^{-\frac{1}{2}}\right)^{n-3} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) - \frac{\left(-t^{\frac{10-3n}{2}}\right) \{1 - (-t)^{n-3}\}}{1+t} \\ & \quad + t^{-\frac{3}{2}} \left(-t^{-\frac{1}{2}}\right)^{n-4} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) + \frac{\left(-t^{\frac{10-3n}{2}}\right) \{1 - (-t)^{n-4}\}}{1+t} \\ &= (-1)^{n-2} t^{\frac{3-n}{2}} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) + (-1)^{n-4} t^{\frac{1-n}{2}} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) + (-1)^{n-4} t^{\frac{2-n}{2}} \\ &= (-1)^{n+1} \left(t^{\frac{4-n}{2}} + t^{\frac{2-n}{2}} + t^{-\frac{n}{2}}\right) \\ &= (-1)^{n+1} t^{-\frac{n}{2}} (t^2 + t + 1). \end{aligned}$$

By (3.11) and (3.12), we have the desired conclusion. \square

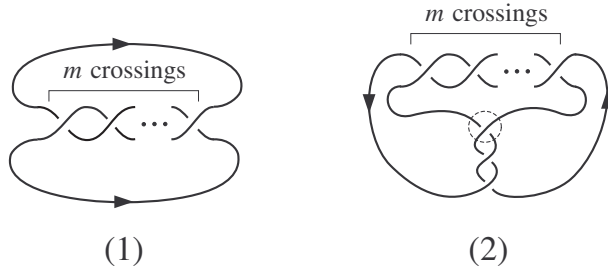


FIGURE 3.6. (1) $(2, -m)$ -torus knot or link N_m , (2) A link ambient isotopic to $J_n[1, \dots, 1, 1]$ if $m = n - 4$

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